

COVARIANT DEFORMED OSCILLATOR ALGEBRAS

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Abstract

The general form and associativity conditions of deformed oscillator algebras are reviewed. It is shown how the latter can be fulfilled in terms of a solution of the Yang-Baxter equation when this solution has three distinct eigenvalues and satisfies a Birman-Wenzl-Murakami condition. As an example, an $SU_q(n) \times SU_q(m)$ -covariant q -bosonic algebra is discussed in some details.

1 Introduction

Since the advent of quantum groups and q -algebras (see e.g. [1] and references quoted therein), much attention has been paid to deformations of the algebras of bosonic and fermionic creation and annihilation operators [2]–[6]. Different deformations of the latter arise depending on which property of the undeformed operators is preserved.

In the simple case of the $su(2)$ Lie algebra, two pairs of bosonic creation and annihilation operators $a_i^\dagger, a_i, i = 1, 2$, give rise to the Jordan-Schwinger realization

$$J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_0 = \frac{1}{2}(N_1 - N_2), \quad (1)$$

where $N_i = a_i^\dagger a_i, i = 1, 2$, are number operators. In addition, the creation operators a_1^\dagger, a_2^\dagger (as well as the modified annihilation operators $\tilde{a}_1 = a_2, \tilde{a}_2 = -a_1$) are the components $+1/2$ and $-1/2$ of an $su(2)$ spinor, respectively. When extending these two properties to the corresponding q -algebra $su_q(2)$ (where q is real and positive), one gets two different sets of q -bosonic operators.

On the one hand, those first considered by Biedenharn [2], Macfarlane [3], Sun and Fu [4], give rise to a Jordan-Schwinger realization of $su_q(2)$ of the same type as (1), where $a_i^\dagger, a_i, i = 1, 2$, now satisfy the relations

$$a_i a_i^\dagger - q^{\pm 1} a_i^\dagger a_i = q^{\mp N_i}, \quad (2)$$

while operators with different indices do still commute, and $a_i^\dagger a_i = [N_i]_q \equiv (q^{N_i} - q^{-N_i})/(q - q^{-1})$. However, the operators a_1^\dagger, a_2^\dagger do not transform any more under a definite representation of the algebra.

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On the other hand, the operators $A_i^\dagger, A_i, i = 1, 2$, introduced by Pusz and Woronowicz [5], satisfy different relations

$$\begin{aligned} A_i^\dagger A_j^\dagger - q^{-1} A_j^\dagger A_i^\dagger &= A_i A_j - q A_j A_i = 0, & i < j, \\ A_i A_j^\dagger - q A_j^\dagger A_i &= 0, & i \neq j, \\ A_i A_i^\dagger - q^2 A_i^\dagger A_i &= I + (q^2 - 1) \sum_{j=1}^{i-1} A_j^\dagger A_j, \end{aligned} \quad (3)$$

where the two modes are not independent any more. As a result of this coupling, the operators A_1^\dagger, A_2^\dagger (as well as $\tilde{A}_1 = q^{1/2} A_2, \tilde{A}_2 = -q^{-1/2} A_1$) are the components $+1/2$ and $-1/2$ of an $su_q(2)$ spinor respectively, but yield an $su_q(2)$ realization that is substantially more complicated than (1). The algebra (3) has also important covariance properties under the quantum group $SU_q(2)$, dual to $su_q(2)$.

The present communication is concerned with the construction of covariant deformed oscillator algebras that generalize the Pusz-Woronowicz algebra for other quantum groups than $SU_q(2)$ (or more generally $SU_q(n)$). The method used will be based on an R -matrix approach similar to that applied in noncommutative differential geometry [7,8]. In Sec. 2, after reviewing the general form and associativity conditions of deformed oscillator algebras, we will show how to fulfil the latter in terms of a solution of the Yang-Baxter equation with three distinct eigenvalues. The example of an $SU_q(n) \times SU_q(m)$ -covariant q -bosonic algebra $\mathcal{A}_q(n, m)$ will be treated in some details in Sec. 3. Finally, in Sec. 4, an alternative derivation of the same algebra, based upon the q -algebra $u_q(n) + u_q(m)$ will be presented.

2 Deformed Oscillator Algebras

Let us consider the complex algebras generated by $I, A_i^\dagger, A_i = (A_i^\dagger)^\dagger, i = 1, \dots, N$, subject to the relations [9,10]

$$\begin{aligned} A_i^\dagger A_j^\dagger &= X_{ij,kl} A_k^\dagger A_l^\dagger, \\ A_i A_j &= X_{ji,lk}^* A_k A_l, \\ A_i A_j^\dagger &= \delta_{ij} + Z_{jl,ik} A_k^\dagger A_l, \end{aligned} \quad (4)$$

where X and Z are some complex $N^2 \times N^2$ matrices, and there are summations over dummy indices. As a consequence of the Hermiticity properties of the generators, X^* is the complex conjugate of X , and Z is a Hermitian matrix.

For these algebras to be associative, it is sufficient to require the braid transposition schemes for triples of generators. The braid scheme for $A_i^\dagger A_j^\dagger A_k^\dagger$ yields the condition

$$X_{ij,ab} X_{bk,cz} X_{ac,xy} = X_{jk,ab} X_{ia,xc} X_{cb,yz}, \quad (5)$$

i.e., in compact tensor notation, the Yang-Baxter equation for X (in the "braid" version)

$$X_{12} X_{23} X_{12} = X_{23} X_{12} X_{23}. \quad (6)$$

Similarly, for $A_i A_j^\dagger A_k^\dagger$, one gets the two conditions

$$\delta_{ji}\delta_{kx} - X_{jk,ix} + Z_{jk,ix} - X_{jk,ab}Z_{ab,ix} = 0, \quad (7)$$

and

$$Z_{kz,ac}Z_{ja,ib}X_{bc,xy} = X_{jk,ab}Z_{bz,cy}Z_{ac,ix}, \quad (8)$$

which may be written in compact form as

$$(I_{12} - X_{12})(I_{12} + Z_{12}) = 0, \quad (9)$$

and

$$Z_{23}Z_{12}X_{23} = X_{12}Z_{23}Z_{12}. \quad (10)$$

From the Hermiticity properties of the generators, it follows that the remaining two triple products $A_i A_j A_k$ and $A_i A_j A_k^\dagger$ will be associative if $A_i^\dagger A_j^\dagger A_k^\dagger$ and $A_i A_j A_k$ are so. Hence, eqs. (6), (9), and (10) are the only associativity conditions of algebra (4).

Let now R be any $N^2 \times N^2$ solution of the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \quad (11)$$

Then the corresponding braid matrix $\hat{R} = \tau R$, where τ is the twist operator (i.e., $\tau_{ij,kl} = \delta_{il}\delta_{jk}$), satisfies an equation similar to (6).

If \hat{R} has three distinct eigenvalues λ_α , $\alpha = 1, 2, 3$, and satisfies a Birman-Wenzl-Murakami (BWM) condition²

$$(\hat{R} - \lambda_1 I)(\hat{R} - \lambda_2 I)(\hat{R} - \lambda_3 I) = 0, \quad (12)$$

then with each eigenspace of \hat{R} , one can associate two solutions of the set of associativity conditions (6), (9), and (10). In terms of the projector

$$\mathcal{P}_\alpha = \prod_{\beta \neq \alpha} \frac{(\hat{R} - \lambda_\beta I)}{(\lambda_\alpha - \lambda_\beta)} \quad (13)$$

onto the eigenspace corresponding to the eigenvalue λ_α , these two solutions can be written as

$$I - X \simeq \mathcal{P}_\alpha \quad \text{and} \quad Z = -\lambda_\alpha^{-1} \hat{R} \quad \text{or} \quad Z = -\lambda_\alpha \hat{R}^{-1}. \quad (14)$$

Considering for instance $Z = -\lambda_\alpha^{-1} \hat{R}$ leads to the following deformed oscillator algebra (written in compact tensor form)

$$A_2^\dagger A_1^\dagger = S A_1^\dagger A_2^\dagger, \quad A_1 A_2 = S^* A_2 A_1, \quad A_1 A_2^\dagger = I_{12} - \lambda_\alpha^{-1} R^{t_1} A_2^\dagger A_1, \quad (15)$$

where $S = \tau X$ is found from (13) and (14), and t_1 means transposition with respect to the first space in the tensor product.

Several examples of such deformed oscillator algebras have been worked out so far [9]–[11]. In all cases, the solution of the Yang-Baxter equation that has been considered is the fundamental R -matrix of some classical quantum group. In such circumstances, the deformed oscillator algebras

²The $SU_q(n)$ -covariant algebra constructed by Pusz and Woronowicz [5] corresponds to the simpler case where \hat{R} has only two distinct eigenvalues, and satisfies a Hecke condition (see Sec. 3).

are left invariant under the transformations induced by the quantum group. The construction presented here is not restricted however to such a choice, and any solution of (11) and (12) might actually be used. In a similar way, deformed oscillator algebras differing from that of Pusz-Woronowicz have been built by considering non-standard solutions of the Yang-Baxter equation and the Hecke condition [12].

The algebras constructed in refs. [9]–[11] include both standard and non-standard ones. The former [9,10] are either of q -bosonic or q -fermionic type, meaning that whenever $q \rightarrow 1$, they go over smoothly into ordinary bosonic or fermionic algebras, respectively. The latter [11], on the contrary, do not have such a smooth behaviour, but share instead some features with the quon algebra [13]. In the next section, we shall consider in more details a covariant q -bosonic algebra generalizing that of Pusz-Woronowicz.

3 An $SU_q(n) \times SU_q(m)$ -Covariant q -Bosonic Algebra

The $SU_q(n)$ quantum group [1] is a complex associative algebra generated by I and the noncommutative elements T_{ij} , $i, j = 1, \dots, n$ of an $n \times n$ matrix T , subject to the relations

$$RT_1T_2 = T_2T_1R, \quad \det_q T = 1, \quad (16)$$

and the $*$ -involution condition

$$T^* = (T^{-1})^t, \quad (17)$$

with q real. In (16), \det_q denotes the quantum determinant, and R is the fundamental R -matrix associated with the A_{n-1} series of Lie algebras,

$$R = q \sum_{i=1}^n e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1 \\ i \neq j}}^n e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{\substack{i,j=1 \\ i < j}}^n e_{ij} \otimes e_{ji}, \quad (18)$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The coproduct, counit and antipode are defined by

$$\Delta(T) = T_1 \dot{\otimes} T_2, \quad \epsilon(T) = 1, \quad S(T) = T^{-1}, \quad (19)$$

where $\Delta(T_{ij}) = T_{ik} \otimes T_{kj}$.

The braid matrix \hat{R} , corresponding to (18), is a real symmetric matrix with two distinct eigenvalues, q and $-q^{-1}$. Their respective multiplicities are $\frac{1}{2}n(n+1)$ and $\frac{1}{2}n(n-1)$, i.e., the dimensions of the symmetric and antisymmetric irreps $[2\dot{0}]_n$ and $[1^2\dot{0}]_n$ of $SU_q(n)$. The \hat{R} -matrix satisfies the Hecke condition

$$(\hat{R} - qI)(\hat{R} + q^{-1}I) = 0. \quad (20)$$

Similar relations are valid for $SU_q(m)$. Its generators and fundamental R -matrix will be denoted by \mathcal{T}_{st} , $s, t = 1, \dots, m$, and \mathcal{R} , respectively, to distinguish them from the corresponding quantities for $SU_q(n)$. Note that T_{ij} and \mathcal{T}_{st} are assumed to commute with one another.

For the product $SU_q(n) \times SU_q(m)$, one can introduce a “large” R -matrix, $\mathbf{R} = q^{-1}R\mathcal{R}$, of dimension $(nm)^2 \times (nm)^2$ [10]. Its matrix elements are defined by

$$\mathbf{R}_{(is)(jt),(ku)(lv)} = q^{-1}R_{ij,kl}\mathcal{R}_{st,uv}. \quad (21)$$

From the properties of the two “small” braid matrices \hat{R} and $\hat{\mathcal{R}}$, it follows that $\hat{\mathbf{R}} = q^{-1}\hat{R}\hat{\mathcal{R}}$ has three distinct eigenvalues q , $-q^{-1}$, and q^{-3} , with respective multiplicities corresponding to the dimensions of the representations $[2\dot{0}]_n[2\dot{0}]_m$, $[2\dot{0}]_n[1^2\dot{0}]_m + [1^2\dot{0}]_n[2\dot{0}]_m$, and $[1^2\dot{0}]_n[1^2\dot{0}]_m$ of $SU_q(n) \times SU_q(m)$, and satisfies the BWM condition (12).

By applying the results of the previous section to the antisymmetric (reducible) eigenspace of $\hat{\mathbf{R}}$ associated with the eigenvalue $-q^{-1}$, one gets a deformed oscillator algebra of type (15), which will be denoted by $\mathcal{A}_q(n, m)$, and whose defining relations are [10]

$$\mathbf{A}_2^\dagger \mathbf{A}_1^\dagger = \mathbf{S} \mathbf{A}_1^\dagger \mathbf{A}_2^\dagger, \quad \mathbf{A}_2 \mathbf{A}_1 = \mathbf{A}_1 \mathbf{A}_2 \mathbf{S}, \quad \mathbf{A}_2 \mathbf{A}_1^\dagger = \mathbf{I}_{21} + q \mathbf{R}^{t_1} \mathbf{A}_1^\dagger \mathbf{A}_2, \quad (22)$$

where

$$\mathbf{S} = \tau(\mathbf{I} - (q + q^{-1})\mathcal{P}_A), \quad \mathcal{P}_A = \frac{(\hat{\mathbf{R}} - q\mathbf{I})(\hat{\mathbf{R}} - q^{-3}\mathbf{I})}{(q + q^{-1})(q^{-1} + q^{-3})}, \quad (23)$$

and the creation and annihilation operators \mathbf{A}_{is}^\dagger , \mathbf{A}_{is} now have two indices, $i = 1, 2, \dots, n$, and $s = 1, 2, \dots, m$. Whenever $q \rightarrow 1$, \mathbf{R} and \mathbf{S} go over into \mathbf{I} , so that (22) becomes an ordinary bosonic algebra.

The defining relations (22) of the q -bosonic algebra $\mathcal{A}_q(n, m)$ may be rewritten in terms of the two “small” R -matrices as

$$R \mathbf{A}_1^\dagger \mathbf{A}_2^\dagger = \mathbf{A}_2^\dagger \mathbf{A}_1^\dagger \mathcal{R}, \quad R \mathbf{A}_2 \mathbf{A}_1 = \mathbf{A}_1 \mathbf{A}_2 \mathcal{R}, \quad \mathbf{A}_2 \mathbf{A}_1^\dagger = \mathbf{I}_{21} \mathcal{I}_{21} + R^{t_1} \mathcal{R}^{t_1} \mathbf{A}_1^\dagger \mathbf{A}_2, \quad (24)$$

or, in a more explicit form, as

$$\begin{aligned} R_{ij,kl} \mathbf{A}_{ks}^\dagger \mathbf{A}_{lt}^\dagger &= \mathbf{A}_{ju}^\dagger \mathbf{A}_{iv}^\dagger \mathcal{R}_{uv,st}, \\ R_{ij,kl} \mathbf{A}_{lt} \mathbf{A}_{ks} &= \mathbf{A}_{iu} \mathbf{A}_{ju} \mathcal{R}_{uv,st}, \\ \mathbf{A}_{is} \mathbf{A}_{jt}^\dagger &= \delta_{ij} \delta_{st} + R_{ki,jl} \mathcal{R}_{us,tv} \mathbf{A}_{ku}^\dagger \mathbf{A}_{lv}. \end{aligned} \quad (25)$$

Let us consider the map $\varphi : \mathcal{A}_q(n, m) \rightarrow \mathcal{A}_q(n, m) \otimes (SU_q(n) \times SU_q(m))$, defined by

$$\begin{aligned} \mathbf{A}_{is}^{\prime\dagger} &= \varphi(\mathbf{A}_{is}^\dagger) = \mathbf{A}_{jt}^\dagger T_{ji} T_{ts}, \\ \mathbf{A}_{is}' &= \varphi(\mathbf{A}_{is}) = \mathbf{A}_{jt} T_{ji}^* T_{ts}^* = T_{ij}^{-1} T_{st}^{-1} \mathbf{A}_{jt}, \end{aligned} \quad (26)$$

where the elements T_{ij} and T_{st} of $SU_q(n) \times SU_q(m)$ are assumed to commute with \mathbf{A}_{is}^\dagger and \mathbf{A}_{is} . As a consequence of (16) and its counterpart for $SU_q(m)$, this map leaves the defining relations (25) of $\mathcal{A}_q(n, m)$ invariant. Hence, the latter is an $SU_q(n) \times SU_q(m)$ -covariant algebra.

In the next section, an important part will be played by the modified annihilation operators

$$\tilde{\mathbf{A}}_{is} = \mathbf{A}_{jt} C_{ji} C_{ts}, \quad C_{ji} = (-1)^{n-j} q^{-(n-2j+1)/2} \delta_{j,i'}, \quad C_{ts} = (-1)^{m-t} q^{-(m-2t+1)/2} \delta_{t,s'}, \quad (27)$$

where $i' \equiv n + 1 - i$, $s' \equiv m + 1 - s$. Eq. (24) can be rewritten in terms of \mathbf{A}_{is}^\dagger , $\tilde{\mathbf{A}}_{is}$ as

$$R \mathbf{A}_1^\dagger \mathbf{A}_2^\dagger = \mathbf{A}_2^\dagger \mathbf{A}_1^\dagger \mathcal{R}, \quad R \tilde{\mathbf{A}}_1 \tilde{\mathbf{A}}_2 = \tilde{\mathbf{A}}_2 \tilde{\mathbf{A}}_1 \mathcal{R}, \quad \tilde{\mathbf{A}}_2 \mathbf{A}_1^\dagger = C_{12} C_{12} + q^2 \mathbf{A}_1^\dagger \tilde{\mathbf{A}}_2 \tilde{R}^{-1} \tilde{R}^{-1}, \quad (28)$$

where \tilde{R} is defined by

$$\tilde{R} = \sum_{i=1}^n e_{ii} \otimes e_{i'i'} + q \sum_{\substack{i,j=1 \\ i \neq j'}}^n e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{\substack{i,j=1 \\ i < j}}^n (-q)^{i-j+1} e_{ij} \otimes e_{i'j'}, \quad (29)$$

and a similar definition holds for $\tilde{\mathcal{R}}$. Under map φ of eq. (26), $\tilde{\mathbf{A}}_{is}$ is transformed into

$$\tilde{\mathbf{A}}'_{is} = \varphi(\tilde{\mathbf{A}}_{is}) = \tilde{\mathbf{A}}_{jt} \tilde{T}_{ji} \tilde{T}_{ts}, \quad \tilde{T} \equiv C^{-1}(T^{-1})^t C, \quad \tilde{\mathcal{T}} \equiv C^{-1}(\mathcal{T}^{-1})^t \mathcal{C}. \quad (30)$$

Finally, combining eqs. (18) and (25) yields the detailed form of the $\mathcal{A}_q(n, m)$ defining relations

$$\begin{aligned} \mathbf{A}_{is}^\dagger \mathbf{A}_{it}^\dagger - q^{-1} \mathbf{A}_{it}^\dagger \mathbf{A}_{is}^\dagger &= 0, & s < t, \\ \mathbf{A}_{is}^\dagger \mathbf{A}_{js}^\dagger - q^{-1} \mathbf{A}_{js}^\dagger \mathbf{A}_{is}^\dagger &= 0, & i < j, \\ \mathbf{A}_{is}^\dagger \mathbf{A}_{jt}^\dagger - \mathbf{A}_{jt}^\dagger \mathbf{A}_{is}^\dagger &= 0, & i > j, \quad s < t, \\ \mathbf{A}_{is}^\dagger \mathbf{A}_{jt}^\dagger - \mathbf{A}_{jt}^\dagger \mathbf{A}_{is}^\dagger &= -(q - q^{-1}) \mathbf{A}_{js}^\dagger \mathbf{A}_{it}^\dagger, & i < j, \quad s < t, \end{aligned} \quad (31)$$

and

$$\begin{aligned} \mathbf{A}_{is} \mathbf{A}_{jt}^\dagger - \mathbf{A}_{jt}^\dagger \mathbf{A}_{is} &= 0, & i \neq j, \quad s \neq t, \\ \mathbf{A}_{is} \mathbf{A}_{js}^\dagger - q \mathbf{A}_{js}^\dagger \mathbf{A}_{is} &= (q - q^{-1}) \sum_{t=1}^{s-1} \mathbf{A}_{jt}^\dagger \mathbf{A}_{it}, & i \neq j, \\ \mathbf{A}_{is} \mathbf{A}_{it}^\dagger - q \mathbf{A}_{it}^\dagger \mathbf{A}_{is} &= (q - q^{-1}) \sum_{j=1}^{i-1} \mathbf{A}_{jt}^\dagger \mathbf{A}_{js}, & s \neq t, \\ \mathbf{A}_{is} \mathbf{A}_{is}^\dagger - q^2 \mathbf{A}_{is}^\dagger \mathbf{A}_{is} &= I + (q^2 - 1) \left(\sum_{j=1}^{i-1} \mathbf{A}_{js}^\dagger \mathbf{A}_{js} + \sum_{t=1}^{s-1} \mathbf{A}_{it}^\dagger \mathbf{A}_{it} \right. \\ &\quad \left. - (q^{-2} - 1) \sum_{j=1}^{i-1} \sum_{t=1}^{s-1} \mathbf{A}_{jt}^\dagger \mathbf{A}_{jt} \right), \end{aligned} \quad (32)$$

together with the Hermitian conjugates of (31). Whenever $m = 1$, substituting $\mathbf{A}_i^\dagger, |\mathbf{A}_i$ for $\mathbf{A}_{i1}^\dagger, \mathbf{A}_{i1}$ in (31) and (32) yields the Pusz-Woronowicz relations (3) for arbitrary n values. Hence, the covariant q -bosonic algebra $\mathcal{A}_q(n, m)$ is a generalization of that of Pusz-Woronowicz for values of m greater than 1.

4 Alternative Derivation in Terms of $u_q(n) + u_q(m)$

An alternative approach to the construction of covariant deformed oscillator algebras, based upon q -algebras, has been developed elsewhere [14,15]. In the case of the algebra $\mathcal{A}_q(n, m)$ introduced in the previous section, one considers the q -algebra $u_q(n) + u_q(m)$. The Cartan-Chevalley generators of $u_q(n)$ are denoted by $E_{ii} = (E_{ii})^\dagger$, $i = 1, 2, \dots, n$, $E_{i,i+1}$, $E_{i+1,i} = (E_{i,i+1})^\dagger$, $i = 1, 2, \dots, n-1$, and satisfy the commutation relations

$$\begin{aligned} [E_{ii}, E_{jj}] &= 0, & [E_{ii}, E_{j,j+1}] &= (\delta_{ij} - \delta_{i,j+1}) E_{j,j+1}, \\ [E_{ii}, E_{j+1,j}] &= (\delta_{i,j+1} - \delta_{ij}) E_{j+1,j}, & [E_{i,i+1}, E_{j+1,j}] &= \delta_{ij} [H_i]_q, \end{aligned} \quad (33)$$

together with the quadratic and cubic q -Serre relations. In (33), $H_i \equiv E_{ii} - E_{i+1,i+1}$. The algebra is endowed with a Hopf algebra structure with coproduct Δ , counit ϵ , and antipode S , defined by

$$\Delta(E_{ii}) = E_{ii} \otimes I + I \otimes E_{ii}, \quad \Delta(E_{i,i+1}) = E_{i,i+1} \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_{i,i+1},$$

$$\Delta(E_{i+1,i}) = E_{i+1,i} \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_{i+1,i}, \quad (34)$$

$$\epsilon(E_{ii}) = \epsilon(E_{i,i+1}) = \epsilon(E_{i+1,i}) = 0, \quad (35)$$

$$S(E_{ii}) = -E_{ii}, \quad S(E_{i,i+1}) = -qE_{i,i+1}, \quad S(E_{i+1,i}) = -q^{-1}E_{i+1,i}. \quad (36)$$

The Cartan-Chevalley generators of $u_q(m)$ are denoted by \mathcal{E}_{ss} , $s = 1, 2, \dots, m$, $\mathcal{E}_{s,s+1}$, $\mathcal{E}_{s+1,s}$, $s = 1, 2, \dots, m-1$, and satisfy relations similar to (33)–(36), while commuting with the generators of $u_q(n)$.

In the approach based upon $u_q(n) + u_q(m)$, the q -bosonic creation operators \mathbf{A}_{is}^\dagger , $i = 1, 2, \dots, n$, $s = 1, 2, \dots, m$, belonging to $\mathcal{A}_q(n, m)$, are defined as the components of a double irreducible tensor $T^{[1^0]_n[1^0]_m}$ with respect to this q -algebra. This means that they fulfil the relations

$$E_{jj}(\mathbf{A}_{is}^\dagger) = \delta_{ji} \mathbf{A}_{is}^\dagger, \quad E_{j,j+1}(\mathbf{A}_{is}^\dagger) = \delta_{j,i-1} \mathbf{A}_{i-1,s}^\dagger, \quad E_{j+1,j}(\mathbf{A}_{is}^\dagger) = \delta_{ji} \mathbf{A}_{i+1,s}^\dagger, \quad (37)$$

$$\mathcal{E}_{tt}(\mathbf{A}_{is}^\dagger) = \delta_{ts} \mathbf{A}_{is}^\dagger, \quad \mathcal{E}_{t,t+1}(\mathbf{A}_{is}^\dagger) = \delta_{t,s-1} \mathbf{A}_{i,s-1}^\dagger, \quad \mathcal{E}_{t+1,t}(\mathbf{A}_{is}^\dagger) = \delta_{ts} \mathbf{A}_{i,s+1}^\dagger, \quad (38)$$

where, for any $u_q(n) + u_q(m)$ generator X , $X(\mathbf{A}_{is}^\dagger)$ denotes the quantum adjoint action $X(\mathbf{A}_{is}^\dagger) = \sum_r X_r^\dagger \mathbf{A}_{is}^\dagger S(X_r^\dagger)$, with $\Delta(X) = \sum_r X_r^\dagger \otimes X_r^\dagger$. The modified annihilation operators $\tilde{\mathbf{A}}_{is}$, $i = 1, 2, \dots, n$, $s = 1, 2, \dots, m$, of eq. (27), are similarly defined as the components of a double irreducible tensor $T^{[0^-]_n[0^-]_m}$ with respect to $u_q(n) + u_q(m)$, and satisfy the relations

$$E_{jj}(\tilde{\mathbf{A}}_{is}) = -\delta_{ji} \tilde{\mathbf{A}}_{is}, \quad E_{j,j+1}(\tilde{\mathbf{A}}_{is}) = \delta_{j,i-1} \tilde{\mathbf{A}}_{i-1,s}, \quad E_{j+1,j}(\tilde{\mathbf{A}}_{is}) = \delta_{ji} \tilde{\mathbf{A}}_{i+1,s}, \quad (39)$$

$$\mathcal{E}_{tt}(\tilde{\mathbf{A}}_{is}) = -\delta_{ts} \tilde{\mathbf{A}}_{is}, \quad \mathcal{E}_{t,t+1}(\tilde{\mathbf{A}}_{is}) = \delta_{ts} \tilde{\mathbf{A}}_{i,s-1}, \quad \mathcal{E}_{t+1,t}(\tilde{\mathbf{A}}_{is}) = \delta_{ts} \tilde{\mathbf{A}}_{i,s+1}. \quad (40)$$

The operators \mathbf{A}_{is}^\dagger and $\tilde{\mathbf{A}}_{is}$ can be explicitly written down in terms of m independent copies of the Pusz-Woronowicz operators [14]. By using such expressions and exploiting the tensorial character of the operators, it is straightforward to prove that their q -commutation relations are given in coupled form by

$$\begin{aligned} [\mathbf{A}^\dagger, \mathbf{A}^\dagger]^{[2^0]_n[1^2]_m} &= [\mathbf{A}^\dagger, \mathbf{A}^\dagger]^{[1^2]_n[2^0]_m} = [\tilde{\mathbf{A}}, \tilde{\mathbf{A}}]^{[0^-]_n[0^-]_m} = [\tilde{\mathbf{A}}, \tilde{\mathbf{A}}]^{[0^-]_n[0^-]_m} = 0, \\ [\tilde{\mathbf{A}}, \mathbf{A}^\dagger]^{[1^0]_n[1^0]_m} &= [\tilde{\mathbf{A}}, \mathbf{A}^\dagger]^{[1^0]_n[1^0]_m} = [\tilde{\mathbf{A}}, \mathbf{A}^\dagger]^{[0^-]_n[1^0]_m} = 0, \\ [\tilde{\mathbf{A}}, \mathbf{A}^\dagger]^{[0^-]_n[0^-]_m} &= \sqrt{[n]_q[m]_q}, \end{aligned} \quad (41)$$

where, for simplicity's sake, the row labels of the coupled $u_q(n) + u_q(m)$ irreps have been dropped. In (41), the coupled q -commutator of two double irreducible tensors $T^{[\lambda_1]_n[\lambda_2]_m}$ and $U^{[\lambda'_1]_n[\lambda'_2]_m}$ is defined by [14]

$$\begin{aligned} &[T^{[\lambda_1]_n[\lambda_2]_m}, U^{[\lambda'_1]_n[\lambda'_2]_m}]_{(M_1)_n(M_2)_m q^\alpha}^{[\Lambda_1]_n[\Lambda_2]_m} \\ &= [T^{[\lambda_1]_n[\lambda_2]_m} \times U^{[\lambda'_1]_n[\lambda'_2]_m}]_{(M_1)_n(M_2)_m}^{[\Lambda_1]_n[\Lambda_2]_m} - (-1)^\epsilon q^\alpha [U^{[\lambda'_1]_n[\lambda'_2]_m} \times T^{[\lambda_1]_n[\lambda_2]_m}]_{(M_1)_n(M_2)_m}^{[\Lambda_1]_n[\Lambda_2]_m}. \end{aligned} \quad (42)$$

Here

$$\begin{aligned} \epsilon &= \phi([\lambda_1]_n) + \phi([\lambda'_1]_n) - \phi([\Lambda_1]_n) + \phi([\lambda_2]_m) + \phi([\lambda'_2]_m) - \phi([\Lambda_2]_m), \\ \phi([\lambda_1]_n) &= \frac{1}{2} \sum_{i=1}^n (n+1-2i) \lambda_{1i}, \quad \phi([\lambda_2]_m) = \frac{1}{2} \sum_{s=1}^m (m+1-2s) \lambda_{2s}, \end{aligned} \quad (43)$$

and

$$\begin{aligned}
& \left[T^{[\lambda_1]_n[\lambda_2]_m} \times U^{[\lambda'_1]_n[\lambda'_2]_m} \right]_{(M_1)_n(M_2)_m}^{[\Lambda_1]_n[\Lambda_2]_m} \\
&= \sum_{(\mu_1)_n(\mu_2)_m(\mu'_1)_n(\mu'_2)_m} \langle [\lambda_1]_n(\mu_1)_n, [\lambda'_1]_n(\mu'_1)_n | [\Lambda_1]_n(M_1)_n \rangle_q \langle [\lambda_2]_m(\mu_2)_m, [\lambda'_2]_m(\mu'_2)_m | [\Lambda_2]_m(M_2)_m \rangle_q \\
&\quad \times T^{[\lambda_1]_n[\lambda_2]_m}_{(\mu_1)_n(\mu_2)_m} U^{[\lambda'_1]_n[\lambda'_2]_m}_{(\mu'_1)_n(\mu'_2)_m},
\end{aligned} \tag{44}$$

where $\langle \cdot, \cdot \rangle_q$ denotes a $u_q(n)$ or $u_q(m)$ Wigner coefficient.

By using the values of the latter, eq. (41) can be written in an explicit form [14]. The resulting relations coincide with eqs. (31) and (32), thereby proving the equivalence of the two constructions of $\mathcal{A}_q(n, m)$ based upon $SU_q(n) \times SU_q(m)$ and $u_q(n) + u_q(m)$, respectively.

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